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A NOTE ON BALANCED BLOCK DESIGNS

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SUMMARY

A necessary and sufficient condition is obtained for a block design to be general efficiency balanced. It is shown that the Fisher's inequality does not hold for all general efficiency balanced and variance balanced block designs, and the lower bound for Fisher's inequality can be replaced by a more stringent bound for the general efficiency as well as variance balanced designs with varying replications and unequal block sizes.

Keywords : General efficiency balanced designs; Fisher's inequality.

Introduction

Consider a connected block design $D(v, b, \mathbf{r}, \mathbf{k})$ having v treatments arranged in b blocks with the jth block being of size k_j, and the ith treatment being replicated r_i times; $\mathbf{r} = (r_1, \ldots, r_v)'$, $\mathbf{k} = (k_1, \ldots, k_b)'$. Let $\mathrm{N} = (n_{ij})_{v \times b}$ be the incidence matrix of D, where n_{ij} is the number of times treatment i occurs in block j. The matrix $\mathbf{C} = \mathbf{R} - \mathrm{N}\,\mathbf{K}^{-1}\mathrm{N}'$ is known as the C-matrix of the design, where $\mathbf{R} = \text{diag}\ (r_1, \ldots, r_v)$, $\mathrm{K} = \text{diag}\ (k_1, \ldots, k_b)$. A design is said to be connected iff Rank (C) $= v - 1$. Throughout this note we consider only connected block designs.

A block design is variance balanced iff every normalized contrast is estimated with the same variance. It is well known that D is variance balanced iff its **C**-matrix is of the form

$$\mathbf{C} = \rho\,(\mathbf{I}_v - \mathbf{1}\,\mathbf{1}'/v), \tag{1}$$

where ρ is a positive scalar, $\mathbf{I}_v$ is an identity matrix of order v, $\mathbf{1}_x$ is an x-component vector with all elements unity.

A block design is efficiency balanced iff all the treatment contrasts are estimated with the same efficiency factor. A block design D is efficiency-balanced iff the **C**-matrix is of the form

$$\mathbf{C} = \mu\,(\mathbf{R} - \mathbf{r}\,\mathbf{r}'/n), \tag{2}$$

where μ is a positive scalar and $n = \mathbf{r}'\,\mathbf{1}$, the total number of observations.

Das and Ghosh [3] defined a general efficiency-balanced block design. A block design D is general efficiency balanced iff the **C**-matrix is of the form

$$\mathbf{C} = \beta\,(\mathbf{G} - \alpha^{-1}\mathbf{g}\,\mathbf{g}'),$$

where $\mathbf{g} = (g_1, \ldots, g^v)'$, $\mathbf{G} = \text{diag}\,(g_1, \ldots, g^v)$, $\alpha = \mathbf{1}'\,\mathbf{g}$ and β a positive scalar. The elements g_i $(i = 1, \ldots, v)$ are arbitrary. For $\mathbf{g} = \mathbf{r}$, D is efficiency-balanced and for $\mathbf{g} = \mathbf{1}$, D is variance-balanced.

The well-known Fisher's inequality $b \geqslant v$ holds for all balanced incomplete block designs. Atiqullah [1] and Raghavarao [5] proved that for binary and equireplicate variance balanced designs, $b \geqslant v$ holds. Dey [2] showed that the Fisher's inequality holds for all non-orthogonal, equireplicate variance balanced designs. Kageyama and Tsuji [4] proved that $b \geqslant v$ holds for all binary variance balanced designs different from randomized complete block designs. In fact they obtained some more classes of variance balanced designs for which $b \geqslant v$ holds. Saha [6] identified a class of non-trivial variance balanced designs for which $b \geqslant v$ holds. However, Saha [6] realized that $b \geqslant v$ does not hold for all variance balanced designs and $b \geqslant v - 1$ holds for a large class of variance balanced designs.

The purpose of this note is to obtain a lower bound to the number of blocks b of a general efficiency-balanced design from which the existing results on Fisher's inequality can be derived as special cases.

2. The Results

The matrix $\mathbf{P} = \mathbf{N}\,\mathbf{K}^{-1}\mathbf{N}'$ of a general efficiency-balanced design is

$$\begin{aligned}\mathbf{P} &= \mathbf{R} - \beta\,(\mathbf{G} - \mathbf{g}\,\mathbf{g}'/\alpha)\\ &= \Phi + c\,\mathbf{g}\,\mathbf{g}'\end{aligned}$$

$$
= \begin{bmatrix} r_1 - \beta g_1 + \beta g_1^2/\alpha & \beta\, g_1 g_2/\alpha & \dots & \beta\, g_1 g_v/\alpha \\ \beta\, g_1 g_2/\alpha & r_2 - \beta g_2 + \beta g_2^2/\alpha & \dots & \beta\, g_2 g_v/\alpha \\ \vdots & & & \\ \beta\, g_1 g_v/\alpha & \beta\, g_2 g_v/\alpha & \dots & r_v - \beta g^v + \beta g_v^2/\alpha \end{bmatrix}, \tag{4}
$$

where Φ is a diagonal matrix and c is a positive scalar. From (4) it follows that a necessary and sufficient condition for a design to be general efficiency balanced is that the off-diagonal elements of **P** are proportional to the relevant g_i values.

If the rows of **P** of a general efficiency-balanced design given in (4), are linearly independent then $R(\mathbf{P}) = v$, where $R(\cdot)$ denotes the rank. Since $R(\mathbf{P}) = R(\mathbf{N}\,\mathbf{K}^{-1}\mathbf{N}') = R(\mathbf{N})$, it follows that $b \geqslant v$ holds for the general efficiency–balanced designs for which the rows of matrix **P** are linearly independent.

It may appear from **P** given in (4) that $R(\mathbf{P}) = v$ and that the Fisher's inequality holds for all general efficiency–balanced designs. This, however, is not true. Consider a special case where the v treatments can be divided into two disjoint sets containing v_1 and v_2 treatments respectively, $v_1 + v_2 = v$. The first set of v_1 treatments have the same replication r_1 and the second set of v_2 treatments have the same replication r_2 so that $v_1 r_1 + v_2 r_2 = n$. Further all the v_1 treatments in the first set have g-values as g_1 and the second set of v_2 treatments have g-values as g_2, so that $v_1 g_1 + v_2 g_2 = \alpha$. For this situation the matrix **P** in (4) simplifies to

$$
\mathbf{P} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{B} \\ \mathbf{B}' & \mathbf{A}_2 \end{bmatrix}, \tag{5}
$$

where

$$\mathbf{A}_1 = (r_1 - \beta g_1)\,\mathbf{I}_{v_1} + (\beta\, g_1^2/\alpha)\,\mathbf{1}\,\mathbf{1}',$$

$$\mathbf{B} = (\beta\, g_1 g_2/\alpha)\,\mathbf{1}_{v_1}\,\mathbf{1}'_{v_2},$$

$$\mathbf{A}_2 = (r_2 - \beta g_2)\,\mathbf{I}_{v_2} + (\beta g_2^2/\alpha)\,\mathbf{1}\,\mathbf{1}'.$$

We now consider the following cases :

Case I. Let $r_t = \beta\, g_t$ for $t = 1$ or 2. Then $R(\mathbf{P}) = v - v_s + 1$, where $s = 2$ if $t = 1$ and $s = 1$ if $t = 2$, and for this type of general efficiency–

balanced design the Fisher's inequality is replaced by a more stringent inequality $b \geqslant v - v_s + 1$, $s = 1$ or 2.

Example 1. Consider the following general efficiency balanced design with parameters $v = 18$, $b = 17$, $v_1 = 8$, $v_2 = 10$, $\mathbf{r} = (10\ \mathbf{1}'_8,\ 17\ \mathbf{1}'_{10})'$, $\mathbf{k} = (14\ \mathbf{1}'_{14}.\ 18\ \mathbf{1}'_3)'$:

$$
\mathbf{N} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
$$

$$\mathbf{C} = (17/7)\,(\mathbf{G} - \mathbf{g}\,\mathbf{g}'/102),\ \mathbf{g} = (4\ \mathbf{1}'_8,\ 7\ \mathbf{1}'_{10})'.$$

It is easy to verify that $\beta\, g_2 = 17 = r_2$ and $b \geqslant 11\ (= v - v_1 + 1)$ holds here.

Case 2. Let $r_1 = (\theta + 1)\,\beta\, g_1 g_2 v_2/\alpha$ and $r_2 = r_1 v_1/\theta v_2$, In this case the matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} \mathbf{E}_1 & \mathbf{B} \\ \mathbf{B}' & \mathbf{E}_2 \end{bmatrix}. \tag{6}$$

where

$$\mathbf{E}_1 = (\theta\ \beta\ g_1 g_2 v_2/\alpha - \beta g_1^2\ v_1/\alpha)\ \mathbf{I}_{v_1} + (\beta\ g_1^2/\alpha)\ \mathbf{1}\ \mathbf{1}',$$

$$\mathbf{E}_2 = (\beta\ g_1 g_2 v_1/\theta\alpha - \beta g_2^2\ v_2/\alpha)\ \mathbf{I}_{v_2} + (\beta\ g_2^2/\alpha)\ \mathbf{1}\ \mathbf{1}'.$$

From the matrix $\mathbf{P}$ in (6) it is seen that for this type of general efficiency balanced designs,

$$[\mathbf{1}'_{v_1}\ \mathbf{E}_1 : \mathbf{1}'_{v_1}\ \mathbf{B}] = \theta\,[\mathbf{1}'_{v_2}\ \mathbf{B}' : \mathbf{1}'_{v_2}\ \mathbf{E}_2].$$

Consequently, $R(\mathbf{P}) = v - 1$ and it follows that for such designs the Fisher's inequality is replaced by a more stringent inequality $b \geqslant v - 1$.

Example 2. Consider the following general efficiency-balanced design with parameters $v = 5$, $b = 4$, $v_1 = 4$, $v_2 = 1$, $\mathbf{r} = (3\ \mathbf{1}'_4, 4)'$, $\mathbf{k} = 4\ \mathbf{1}_4$:

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{aligned} \mathbf{C} &= (11/8)\,(\mathbf{G} - \mathbf{g}\ \mathbf{g}'/11), \\ \mathbf{g} &= (2\ \mathbf{1}'_4, 3)'. \end{aligned}$$

It is easy to verify that $r_1 = (\theta + 1)\ \beta\ g_1 g_2 v_2/\alpha = 3$ and $r_1/r_2 = 3/4 = \theta v_2/v_1$, $\theta = 3$. Therefore $b \geqslant 4\ (= v - 1)$ holds here.

Some special cases of a general efficiency-balanced design are now considered.

Case 1. For $\mathbf{g} = \mathbf{r}$, the design is efficiency balanced. For $\beta = 1$, the matrix $\mathbf{P}$, given in (4), simplifies to $\mathbf{r}\ \mathbf{r}'/n$ and $R(\mathbf{P}) = 1$. It is well known that an efficiency-balanced design with $\beta = 1$ reduces to an orthogonal design. So barring orthogonal designs, $b \geqslant v$ holds for all efficiency balanced designs.

Case 2. A particular case of interest is when $\mathbf{g} = \mathbf{1}$, for which the matrix $\mathbf{P}$ of (4) is of the form

$$\mathbf{P} = \begin{bmatrix} r_1 - \beta + \beta/v & \beta/v & \cdots & \beta/v \\ \beta/v & r_2 - \beta + \beta/v & \cdots & \beta/v \\ \vdots & \vdots & & \vdots \\ \beta/v & \beta/v & \cdots & r_v - \beta + \beta/v \end{bmatrix},$$

For an equi-replicated design, i.e. $\mathbf{r} = r\,\mathbf{1}$, the rows of $\mathbf{P}$ are linearly independent unless $\beta = r$, in which case $\mathbf{P} = (r/v)\,\mathbf{1}\,\mathbf{1}'$. So, Fisher's inequality holds for all non-orthogonal equi-replicated variance balanced designs.

For any variance balanced design with varying replications let $r_i \neq \beta$ for any $i = 1\,(1)v$. The rows of $\mathbf{P}$ are then linearly independent and $b \geqslant v$ holds for these variance balanced designs. Now let $r_1 = r_2 = \ldots = r_t = \beta$, where $t < v$. In this case the first t rows of $\mathbf{P}$ are identical and $R(\mathbf{P}) = v - t + 1$. For such a design, the Fisher's inequality can be replaced by a more stringent inequality $b \geqslant v - t + 1$. This result has also been proved by Saha [7] in a different way who has identified all variance balanced designs for which $b \geqslant v$ holds. This was pointed out to the authors by the referee.

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